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Abstract. Starting from Wilson's differential renormalisation group equation with smooth momentum cut-off we calculate the spin–spin correlation $\langle S_{\boldsymbol{q}} S_{-\boldsymbol{q}}\rangle$ near the critical point for the one-component vector model in lowest non-trivial order in ϵ. We show that it differs only by a short-range term and a trvial factor from the self-correlation of the wavevector-dependent eigenoperator $O_{\mathrm{h}}(\boldsymbol{q}, g)$. This self-correlation is calculated following the procedure given by Wegner and expanded in expectation value of translationally invariant eigenoperators. Comparisons with the results of various authors are made.

1. Introduction

In this paper we investigate the singular behaviour of the spin–spin correlation function $\langle S_{\boldsymbol{q}} S_{-\boldsymbol{q}}\rangle$ near the normal critical point as $\tau \propto T - T_{\mathrm{c}}$ approaches zero. Brézin *et al* (1974a, b) derived the expression

$$\langle S_{\boldsymbol{q}} S_{-\boldsymbol{q}}\rangle \approx q^{-2+\eta}(A + Bq^{-1/\nu}\tau + C_{\pm}(q^{-1/\nu}|\tau|)^{1-\alpha} + \ldots) \tag{1.1}$$

by means of the Callan–Symanzik equation. A, B, C_+ (for $T > T_{\mathrm{c}}$) and C_- (for $T < T_{\mathrm{c}}$) are constants. α, η and ν are the common critical exponents. The result (1.1) was suggested by Fisher and Langer (1968). Brézin *et al* (1974a, b, 1976) generalised it to the case of a finite magnetic field. Fisher and Aharony (1973) showed that the expression (1.1) is consistent with an expansion of the correlation function around dimensionality four and they determined the coefficients A, B and $C_{\pm}$.

On the basis of a differential renormalisation group (RG) equation by Wilson and Kogut (1974) (see § 2), Wegner (1975, 1976) gave a general procedure (see § 3) for calculating the correlations of eigenoperators and described their singular and scaling structures near criticality. The purpose of this paper is to show how to calculate the spin–spin correlation function in lowest order in $\epsilon = 4 - d$ for the one-component model by means of Wegner's procedure (d is the dimensionality of the system.) Since $S_{\boldsymbol{q}}$ is not an eigenoperator of the linearised RG it has to be expanded in eigenoperators. We found that $S_{\boldsymbol{q}}$ is a linear combination of the two eigenoperators $O_{\mathrm{h}}(\boldsymbol{q})$ and $O_{\mathrm{r}}(\boldsymbol{q})$ for arbitrary translationally invariant Hamiltonians (Stützer 1977). Since O_{r} is redundant the singular contributions of the spin–spin correlation are given only by the singular part of $\langle O_{\mathrm{h}}(\boldsymbol{q}) O_{\mathrm{h}}(-\boldsymbol{q})\rangle$. Combining the scaling laws for this singular part and for the function $(\langle S_{\boldsymbol{q}} S_{-\boldsymbol{q}}\rangle - V)w^2(q)$ given by Wilson and Kogut (1974) in § 4 we derive a general expression for the spin–spin correlation as a function of the wavevector $\boldsymbol{q}$ and the scaling fields g_i where the $\boldsymbol{q}$ dependence can be given explicitly. (V denotes the volume of the system and $w(q)$ is a regular function depending on the cut-off function in the RG equation.) Furthermore we see in § 4 that in the critical limit the spin–spin correlation is an exact homogeneous function of q and g_i and that one can obtain expression (1.1). The first three expansion coefficients in (1.1) were determined by evaluating $\langle O_h(q)O_h(-q)\rangle$ (Stützer 1977). These coefficients do not depend on the cut-off function which is shown in § 5. There we discuss universal ratios, give the results of our calculations and compare them with those of various authors.

2. Renormalisation group equation

The Hamiltonian H of the systems considered here (a factor $-1/k_BT$ is incorporated in H) is a functional of the Fourier components

$$S_q = \int d^d r S(r)\, e^{-iq\cdot r}$$

of a classical one-component spin field $S(r)$ in local space. A translationally invariant Hamiltonian can be written as follows:

$$H = u_0 V + u_1 S_0 + \frac{1}{2!}\int_q u_2(q) S_q S_{-q} + \frac{1}{3!}\int_{q_1 q_2} u_3(q_1, q_2) S_{q_1} S_{q_2} S_{-q_1-q_2} + \dots \tag{2.1}$$

with

$$\int_q \equiv \frac{1}{(2\pi)^d}\int_{-\infty}^{\infty} d^d q.$$

We require the functions u_n to be regular at $q = 0$. Then the interaction is short ranged. In the theory of critical phenomena developed by Wegner (1972) the differential RG equation

$$dH/dl = \mathscr{G}H \tag{2.2}$$

determines the functions $u_n(q_1, \dots, q_{n-1}, l)$ for a given initial Hamiltonian H_0. The formal solution of (2.2) is $H_l = e^{\mathscr{G}l}H_0$ $\mathscr{G}$ is the generator of the RG transformations $\mathscr{R}_l = e^{\mathscr{G}l}$. It is constructed such that the partition function Z is conserved. For our calculations we use the RG generator (Wilson and Kogut 1974)

$$\mathscr{G}H = dV\frac{\partial H}{\partial V} + \int_q (\tfrac{1}{2}dS_q + q\cdot\nabla S_q)\frac{\partial H}{\delta S_q}$$
$$+ \int_q (1 + \tfrac{1}{2}\eta + \theta(q^2))\left(S_q\frac{\delta H}{\delta S_q} + \frac{\delta^2 H}{\delta S_q \delta S_{-q}} - \frac{\delta H}{\delta S_q}\frac{\delta H}{\delta S_{-q}} - 1\right). \tag{2.3}$$

The first term and the first integral of the right-hand side of equation (2.3) change the length scale by a factor e^{-l}, i.e. $q \to q\, e^{l}$. The second integral transforms the variables S_q. This transformation acts like an incomplete elimination of S_q with large q (RG with smooth momentum cut-off). The otherwise arbitrary cut-off function $\theta(q^2)$ in (2.3) with $\theta(0) = 0$ has to be regular in q^2 and monotonically increasing. The constant η is determined such that the requirement (2.4) (see below) can be fulfilled.

H_l depends on a finite number of (l-independent) thermodynamical variables v_i like temperature T, magnetic field h, etc. For the Hamiltonian H_l^c at the critical point, $\{v_i^c\}$, one requires that there is a fixed point H^* such that

$$\lim_{l\to\infty} H_l^c = H^*. \tag{2.4}$$

A fixed-point Hamiltonian obeys

$$\mathrm{d}H^*/\mathrm{d}l = 0; \qquad \mathscr{G}H^* = 0. \tag{2.5}$$

In the following we sketch Wegner's procedure (1972, 1976) of representing a Hamiltonian H_l as a function of l and the v_i. According to this procedure one assumes that there exist parameters g_i, the scaling fields, and constants y_i, the scaling exponents, such that the g_j are functions of the v_i and the Hamiltonian H_l can be expanded in powers of the terms $g_j\,\mathrm{e}^{y_j l}$. We write $g_j(v)$ and

$$H_l = H(g\,\mathrm{e}^{yl}) \tag{2.6}$$

where $g\,\mathrm{e}^{yl}$ and v denote the ordered sets $\{g_i\,\mathrm{e}^{y_i l}\}$ and $\{v_i\}$ respectively. Here the y_j are real numbers. The scaling fields g_j with $y_j > 0$ ($y_j < 0$) are called relevant (irrelevant). It follows from the requirement (2.4) that the relevant scaling fields g_j^{rel} have to vanish at the critical point

$$g_j^{\mathrm{rel}}(v^c) = 0 \tag{2.7}$$

whereas the irrelevant scaling fields g_j^{irr} may be different from zero. Furthermore we must require $H(0) = H^*$.

To find the scaling exponents y_j, the functions $g_j(v)$ and finally $H(g)$ we can proceed as follows. We rewrite the RG equation (2.2) by (2.6):

$$DH(g) = \mathscr{G}H(g) \tag{2.8}$$

with

$$D \equiv \sum_j y_j g_j \frac{\partial}{\partial g_j}. \tag{2.9}$$

Differentiating both sides of equation (2.8) with respect to g_i we obtain from (2.3) the eigenvalue equation

$$\mathscr{L}(g)O_i(g) = y_i O_i(g) \tag{2.10}$$

for the linearised RG generator

$$\mathscr{L}(g) = dV\frac{\partial}{\partial V} + \int_{\boldsymbol{q}} \left[\tfrac{1}{2} dS_{\boldsymbol{q}} + \boldsymbol{q}\cdot\nabla S_{\boldsymbol{q}} + (1 + \tfrac{1}{2}\eta + \theta(q^2))\left(S_{\boldsymbol{q}} + \frac{\delta}{\delta S_{-\boldsymbol{q}}} - 2\frac{\delta H(g)}{\delta S_{-\boldsymbol{q}}}\right)\right]\frac{\delta}{\delta S_{\boldsymbol{q}}} - D$$

with its eigenoperators

$$O_i(g) = \partial H(g)/\partial g_i \tag{2.12}$$

and eigenvalues y_i. Now we are able to calculate the y_i and the eigenoperators $O_i \equiv O_i(0)$ of $\mathscr{L}(0)$ by solving equation (2.10) at $g = \{0\}$. First, however, we must calculate a fixed-point H^* because $\mathscr{L}$ depends on H^*.

The volume V is the eigenoperator O_0 of $\mathscr{L}$ with the scaling exponents $y_0 = d$. We assume that the operators O_i form a complete set in the space of the functions of $S_{\boldsymbol{q}}$. Then it is convenient to expand H_l in the O_i around H^*:

$$H_l = H^* + \sum_i \mu_i(l) O_i. \tag{2.13}$$

Because of the assumption above the functions $\mu_i(l)$ can be expanded as a power series in $g_i\,e^{y_i l}$:

$$\mu_i(l)=\sum_N \frac{1}{N!} b_{iN} g_N\, e^{y_N l} \tag{2.14}$$

where g_N stands for any product $g_{j_1}g_{j_2}\ldots g_{j_r}$ of scaling fields with $N=\{j_1, j_2, \ldots, j_r\}$, $N!=r!$. $g_\varnothing=1$, $y_N=y_{j_1}+y_{j_2}+\ldots+y_{j_r}$ and $y_\varnothing=0$.

Inserting (2.13) into (2.2) one obtains differential equations for the functions $\mu_i(l)$. Since the RG generator is non-linear in H (see (2.3)), these differential equations are non-linear in $\mu_i(l)$ and coupled. Therefore in general it is not easy to find exact solutions. But we can determine the coefficients b_{iN} from (2.14) by iteration—provided, however, the inequality $y_i \neq y_N$ $(N\neq\{i\})$ holds, which we assume. Then an expansion of H_l in the terms $g_j\,e^{y_j l}$ is defined. For the operators $O_j(g)$ we get from (2.12), (2.13) and (2.14)

$$O_j(g)=\sum_{i,N}\frac{1}{N!} b_{iM} g_N O_i \tag{2.15}$$

with $M\equiv N\cup\{i\}$.

Another method is to expand $\Sigma_i\,\mu_i(l)O_i$ in the fields $\mu_i\equiv\mu_i(0)$. From (2.2) and (2.11), in the linear approximation we get

$$H_l=H^*+\sum_i \mu_i\, e^{y_i l}O_i+\mathrm{O}(\mu^2). \tag{2.16}$$

By iteration it is possible to compute the higher-order terms in μ_i. Comparing (2.16) with (2.14) we set $b_{i\varnothing}=0$ and $b_{i\{j\}}=\delta_{ij}$.

Now one requires that each relevant field μ_i^{rel} (with $y_i>0$) in (2.16) depends linearly on this v_i which is thermodynamically conjugate to O_i. According to the requirement (2.4) we write $\mu_i^{\mathrm{rel}}=c_i(v_i-v_i^{\mathrm{c}})$ with an appropriate scale factor c_i. From the inversion of the series (2.14) we can see that then the relevant scaling fields $g_j^{\mathrm{rel}}(v)$ can be expanded in the v_i. Because of (2.7) we have $g_j^{\mathrm{rel}}(v)=c_j(v_j-v_j^{\mathrm{c}})$ in first order. For the temperature difference $\mu_{\mathrm{E}}=c_{\mathrm{E}}(T-T_{\mathrm{c}})$ we introduce

$$\tau\equiv\mu_{\mathrm{E}}, \qquad g_{\mathrm{E}}=\tau+\mathrm{O}(\tau^2) \tag{2.17}$$

with the scaling exponent $y_{\mathrm{E}}>0$ and with the operator

$$O_{\mathrm{E}}(g)=O_{\mathrm{E}}+\sum_i b_{i\{\mathrm{E,E}\}}\tau O_i+\mathrm{O}(\tau^2). \tag{2.18}$$

3. Operator–operator correlation function

In order to compute wavevector-dependent correlation functions on the basis of a RG equation it is convenient to use, in addition to equation (2.10), the eigenvalue equation

$$\mathscr{L}(\boldsymbol{q}, g)O_i(\boldsymbol{q}, g)=y_iO_i(\boldsymbol{q}, g) \tag{3.1}$$

for $\mathscr{L}(\boldsymbol{q}, g)=\mathscr{L}(g)-\boldsymbol{q}\cdot\nabla$ with $\boldsymbol{q}$-dependent short-range eigenoperators $O_i(\boldsymbol{q}, g)$, $O_i(g)=O_i(0, g)$. Wegner (1975, 1976) developed a procedure for calculating operator–operator correlation functions $\langle O_j(\boldsymbol{q}, g)O_k(-\boldsymbol{q}, g)\rangle$ starting from the initial Hamiltonian

$$H_0=H(g)+\kappa_jO_j(\boldsymbol{q}, g)+\kappa_kO_k(-\boldsymbol{q}, g). \tag{3.2}$$

The parameters κ_j and κ_k depend on $\boldsymbol{q}$. From (3.2) one gets the following expression for the correlation functions:

$$\langle O_j(\boldsymbol{q}, g)O_k(-\boldsymbol{q}, g)\rangle = V\frac{\partial^2 F(H_0)}{\partial\kappa_j\,\partial\kappa_k}\bigg|_{\kappa_j=\kappa_k=0} \tag{3.3}$$

where $VF = \ln Z$ is the free energy. A factor $-1/k_\mathrm{B}T$ is incorporated in F.

Now we will find the Hamiltonian H_l by applying the RG equation (2.2). For that purpose we represent H_l as a sum of $H(g\,\mathrm{e}^{yl})$ and an inhomogeneous part which we expand in operators $O_i(\boldsymbol{q}_i\,\mathrm{e}^l, g\,\mathrm{e}^{yl})$ and in powers of κ_j and κ_k. Then we solve (2.2) iteratively and get first the Hamiltonian linear in κ (cf (2.16)):

$$H_l^{\mathrm{lim}} = H(g\,\mathrm{e}^{yl}) + \kappa_j\,\mathrm{e}^{y_jl}O_j(\boldsymbol{q}\,\mathrm{e}^l, g\,\mathrm{e}^{yl}) + \kappa_k\,\mathrm{e}^{y_kl}O_k(-\boldsymbol{q}\,\mathrm{e}^l, g\,\mathrm{e}^{yl}) \tag{3.4}$$

where we use (3.1) and the initial condition (3.2). In the next step the non-linear part of $\mathscr{G}H_l^{\mathrm{lim}}$ generates perturbations of order $\kappa_j\kappa_k$, κ_j^2 and κ_k^2. Here we are interested only in the contribution proportional to $\kappa_j\kappa_k$. This is expanded merely in translational invariant operators $O_i(g\,\mathrm{e}^l) = O_i(\boldsymbol{q}_i\,\mathrm{e}^l, g\,\mathrm{e}^{yl})$ with $\boldsymbol{q}_i = \boldsymbol{q} + (-\boldsymbol{q}) = \boldsymbol{0}$. Thus one gets

$$H_l = H_l^{\mathrm{lim}} + \kappa_j\kappa_k \sum_i \tilde{M}_{ijk}(\boldsymbol{q}, g, l)O_i(g\,\mathrm{e}^{yl}) + \mathrm{O}(\kappa_j^2, \kappa_k^2). \tag{3.5}$$

At this point we assume that the $\boldsymbol{q}_i\,\mathrm{e}^l$-dependent operators in H_l with $\boldsymbol{q}_i \neq \boldsymbol{0}$ become very small if l tends to ∞. This property was not proved generally (cf Stützer 1977), but it is fulfilled in all known cases (Wilson and Kogut 1974, Wegner 1976). Inserting H_l from (3.5) into (2.2) one can derive differential equations for the coefficients $\tilde{M}_{ijk}$. From the general solutions of these differential equations one can see (Stützer 1977) that the limit

$$\lim_{l\to\infty} \mathrm{e}^{-y_il}\tilde{M}_{ijk}(\boldsymbol{q}, g, l) = M_{ijk}(\boldsymbol{q}, g) \tag{3.6}$$

exists (cf Wegner 1975). After neglecting the wavevector-dependent operators in H_l with a sufficiently large $l \geq \tilde{l}$ we integrate the RG equation from $\tilde{l}$ to 0, using (2.10), (3.6) and the initial Hamiltonian $H_{\tilde{l}}$. This yields in linear order in $\kappa_j\kappa_k$

$$\tilde{H}_0 = H(g) + \kappa_j\kappa_k \sum_i M_{ijk}(\boldsymbol{q}, g)O_i(g) + \mathrm{O}(\kappa_j^2, \kappa_k^2). \tag{3.7}$$

From the general solutions of the differential equations for the expansion coefficients M_{ijk} one can show that M_{ijk} consists of two contributions:

$$M_{ijk} = R_{ijk} + S_{ijk}. \tag{3.8}$$

R_{ijk} is regular at $\boldsymbol{q}$ provided none of the numbers $y_i - y_j - y_k - y_N$ are non-negative integers. Otherwise logarithmic singularities arise which we do not discuss here. The function S_{ijk} becomes singular at $\boldsymbol{q} = \boldsymbol{0}$.

From (3.7) we can easily derive

$$V\frac{\partial^2 F(H_0)}{\partial\kappa_j\,\partial\kappa_k}\bigg|_{\kappa_j=\kappa_k=0} = \sum_i M_{ijk}(\boldsymbol{q}, g)\langle O_i(g)\rangle. \tag{3.9}$$

Since the free energy VF is invariant under the RG transformation $H_0 \to \tilde{H}_0$, the following representation for operator–operator correlation functions follows from (3.3), (3.8) and (3.9):

$$\langle O_j(\boldsymbol{q}, g)O_k(-\boldsymbol{q}, g)\rangle = \sum_i (R_{ijk}(\boldsymbol{q}, g) + S_{ijk}(\boldsymbol{q}, g))\langle O_i(g)\rangle. \tag{3.10}$$

4. General formulae for the spin–spin correlation

In general, the operator $S_{\boldsymbol{q}}$ is not an eigenoperator of $\mathscr{L}$. In order to calculate the spin–spin correlation $\langle S_{\boldsymbol{q}}S_{-\boldsymbol{q}}\rangle$ by the RG methods described before we need to expand $S_{\boldsymbol{q}}$ in eigenoperators of $\mathscr{L}$. This is possible (Stützer 1977) because the operators

$$O_{\mathrm{h}}(\boldsymbol{q}, g) = w(q)(S_{\boldsymbol{q}} - \delta H(g)/\delta S_{-\boldsymbol{q}}) \tag{4.1}$$

$$O_{\mathrm{r}}(\boldsymbol{q}, g) = w(q)\delta H(g)/\delta S_{-\boldsymbol{q}} \tag{4.2}$$

are eigenoperators of $\mathscr{L}$ for arbitrary $H(g)$. The function

$$w(q) = \exp\left(\int_0^{q^2} \frac{\theta(x^2)}{x} \mathrm{d}x \right) \tag{4.3}$$

is regular for small q. Thus we get the desired relation

$$S_{\boldsymbol{q}} = w^{-1}(q)(O_{\mathrm{h}}(\boldsymbol{q}, g) + O_{\mathrm{r}}(\boldsymbol{q}, g)) \tag{4.4}$$

and thereby

$$\langle S_{\boldsymbol{q}}S_{-\boldsymbol{q}}\rangle = w^{-2}(g)(\langle O_{\mathrm{h}}(\boldsymbol{q}, g)O_{\mathrm{h}}(-\boldsymbol{q}, g)\rangle + 2\langle O_{\mathrm{h}}(\boldsymbol{q}, g)O_{\mathrm{r}}(-\boldsymbol{q}, g)\rangle$$
$$+\langle O_{\mathrm{r}}(\boldsymbol{q}, g)O_{\mathrm{r}}(-\boldsymbol{q}, g)\rangle). \tag{4.5}$$

Now we make use of the fact that $O_{\mathrm{r}}(\boldsymbol{q}, g)$ is redundant (Stützer 1977). Redundant operators $O_j^{\mathrm{red}}(\boldsymbol{q}, g)$ have the properties (Wegner 1976) that their expectation value vanishes and their correlation with another eigenoperator of $\mathscr{L}$ is short ranged:

$$\langle O_j^{\mathrm{red}}(\boldsymbol{q}, g)O_k(-\boldsymbol{q}, g)\rangle = \sum_i R_{ijk}(\boldsymbol{q}, g)\langle O_i(g)\rangle, \tag{4.6}$$

$Q_{\mathrm{h}}(\boldsymbol{q}, g)$ is non-redundant. Thus the equation

$$\langle S_{\boldsymbol{q}}S_{-\boldsymbol{q}}\rangle = w^{-2}(q)\sum_i (R_i(\boldsymbol{q}, g) + S_i(\boldsymbol{q}, g))\langle O_i(g)\rangle \tag{4.7}$$

follows from (3.10), (4.5) and (4.6) with the abbreviations $R_i \equiv R_{i\mathrm{hh}} + 2R_{i\mathrm{hr}} + R_{i\mathrm{rr}}$ and $S_i \equiv S_{i\mathrm{hh}}$. R_i and S_i are regular in the scaling fields g_i (Wegner 1975). We expand

$$\langle S_{\boldsymbol{q}}S_{-\boldsymbol{q}}\rangle = w^{-2}(g)\sum_{i,N} (r_{iN}(\boldsymbol{q}) + s_{iN}(\boldsymbol{q}))g_N\langle O_i(g)\rangle \tag{4.8}$$

with the notation from § 2. $r_{iN}(\boldsymbol{q})$ is regular at $\boldsymbol{q} = \mathbf{0}$, $s_{iN}(\boldsymbol{q})$ is singular at $\boldsymbol{q} = \mathbf{0}$.

Now it is possible to determine the $\boldsymbol{q}$-dependence of $\langle S_{\boldsymbol{q}}S_{-\boldsymbol{q}}\rangle$ by applying the three scaling laws ,

$$s_{iN}(\boldsymbol{q}) = \mathrm{e}^{(2y_{\mathrm{h}} - y_i)l} s_{iN}(\boldsymbol{q}\,\mathrm{e}^l) \tag{4.9}$$

$$\langle O_i(g)\rangle_0 = \mathrm{e}^{(y_i - d)l}\langle O_i(g\,\mathrm{e}^{yl})\rangle_l \tag{4.10}$$

$$w^2(q)(\langle S_{\boldsymbol{q}}S_{-\boldsymbol{q}}\rangle_0 - V) = \mathrm{e}^{(2-\eta)l}w^2(q\,\mathrm{e}^l)(\langle S_{\boldsymbol{q}\,\mathrm{e}^l}S_{-\boldsymbol{q}\,\mathrm{e}^l}\rangle - V) \tag{4.11}$$

where y_{h} in (4.9) is the positive scaling exponent

$$y_{\mathrm{h}} = \tfrac{1}{2}(d + 2 - \eta) \tag{4.12}$$

of $O_{\mathrm{h}}(\boldsymbol{q}, g)$. The index of $\langle\ldots\rangle_l$ indicates that the expectation value is taken for $H(g\,\mathrm{e}^{yl})$. Equations (4.9) and (4.10) are given by Wegner (1975, 1976). Wilson and Kogut (1974) derived equation (4.11) by means of a different RG generator than that in (2.3) (cf Wegner 1976).

We choose the RG parameter l so that $\boldsymbol{q}\,\mathrm{e}^{l}=\boldsymbol{e}$ with $\boldsymbol{q}=q\boldsymbol{e}$ holds. Then we apply the scaling laws (4.9)–(4.11) to equation (4.8) and get

$$\sum_{i,N} r_{iN}(\boldsymbol{q})g_N\langle O_i(g)\rangle - w^2(q)V$$

$$= q^{-2+\eta}\Big(\sum_{i,N} q^{y_i-y_N-d} r_{iN}(\boldsymbol{e})g_N\langle O_i(g)\rangle - w^2(1)V\Big). \tag{4.13}$$

We assumed above (see § 3) that none of the exponents $y_i-2y_{\mathrm{h}}-y_N$ are non-negative integers. Furthermore we assume that $0\leq\eta\ll 1$ holds. This turns out to hold in most cases. Therefore the left-hand side of (4.13) is singular at $\boldsymbol{q}=0$. Now since the functions r_{iN} and w are regular at $\boldsymbol{q}=0$, both sides of equation (4.13) must vanish. Then we obtain the desired expression

$$\langle S_{\boldsymbol{q}}S_{-\boldsymbol{q}}\rangle = V + w^{-2}(q)q^{-2+\eta}\sum_{i,N} s_{iN}(\boldsymbol{e})q^{y_i-y_N-d}g_N\langle O_i(g)\rangle. \tag{4.14}$$

Now we investigate the spin–spin correlation as a function of the scaling fields. Let us apply the scaling law (4.10) and set $|g_{\mathrm{E}}|\,\mathrm{e}^{y_{\mathrm{E}}l}=1$ (see (2.17)). We assume that $\langle O_i(g)\rangle$ is expendable in the irrelevant scaling fields g_j^{irr} (Wegner 1976). Then we get

$$\langle O_i(g)\rangle = \sum_N |g_{\mathrm{E}}|^{(d-y_i-y_N)/y_{\mathrm{E}}}\frac{1}{N!}f_{iN}(g^{\mathrm{rel}}|g_{\mathrm{E}}|^{-y/y_{\mathrm{E}}})g_N^{\mathrm{irr}} \tag{4.15}$$

with

$$f_{iN}(g^{\mathrm{rel}}) = \frac{\partial}{\partial g_N^{\mathrm{irr}}}\langle O_i(g)\rangle|_{g^{\mathrm{irr}}=\{0\}} \tag{4.16}$$

and with $\partial/\partial g_N=\partial^r/\partial g_{j1}\ldots\partial g_{j_r}$.

On approaching the critical point the relevant scaling fields g_j^{rel} tend to zero because of (2.7), whereas the irrelevant scaling fields g_j^{irr} approach some constants. Let us now study the leading singular behaviour of the spin–spin correlation function in the critical limit $\boldsymbol{q}\to 0$ and $g_j^{\mathrm{rel}}\to 0$. In doing so we apply the transformation $\boldsymbol{q}\to b\boldsymbol{q}$, $g_j^{\mathrm{rel}}\to b^{y_j}g_j^{\mathrm{rel}}$, $g_j^{irr}\to g_j^{\mathrm{iee}}$ to equation (4.14) with (4.15). Then with $b\to 0$ this transformation gives us the correct scaling behaviour prescribed by the RG for the critical limit. Since y_j is positive for relevant and negative for irrelevant scaling fields, we find in the critical limit $b\to 0$ that the terms $|g_{\mathrm{E}}|^{-y_N/y_{\mathrm{E}}}g_N^{\mathrm{irr}}$ and $q^{-y_N}g_N$ (with at least one irrelevant scaling field) vanish whereas the terms $g_j^{\mathrm{rel}}|g_{\mathrm{E}}|^{-y_j/y_{\mathrm{E}}}$, $q^{y_i-d}|g_{\mathrm{E}}|^{(d-y_i)/y_{\mathrm{E}}}$ and $q^{-y_N}g_N^{\mathrm{rel}}$ are constant. The term $q^{-2+\eta}$ diverges and the leading singular behaviour thus reads

$$\langle S_{\boldsymbol{q}}S_{-\boldsymbol{q}}\rangle \approx q^{-2+\eta}\sum_{i,N} s_{iN}(\boldsymbol{e})q^{y_i-y_N-d}g_N^{rel}|g_{\mathrm{E}}|^{(d-y_i)/y_{\mathrm{E}}}\frac{1}{N!}f_{iN}(g^{\mathrm{rel}}|g_{\mathrm{E}}|^{-y/y_{\mathrm{E}}}) \tag{4.17}$$

where we used $w(0)=1$. All operators $O_i(g)$ contribute to the leading singular behaviour. Near the critical point the expectation value $\langle O_i(g)\rangle$ give the singularities of the spin–spin correlation as a function of the thermodynamic variables. The function on the right-hand side of (4.17) obviously obeys scaling. One can easily see that this scaling function represents the leading scaling behaviour of $\Sigma\, S_i(\boldsymbol{q},g)\langle O_i(g)\rangle$ (cf Wegner 1975). The universality of this function was proved (Wegner 1976 and references therein).

In the case of normal critical point there exist the two relevant scaling fields g_{E} and $g_{\mathrm{h}}=c_{\mathrm{h}}h+O(h^2)$ where h denotes the magnetic field. g_{h} has the scaling exponent y_{h} (see (4.12)) and is conjugate to the operator $O_{\mathrm{h}}(\boldsymbol{q},g)$ (see (4.1)). $\langle O_{\mathrm{h}}\rangle=\langle S_0\rangle$ is the magnetisation of the system. Universality means in particular (Wegner 1976) that the leading contributions $S_i(\boldsymbol{q}, g^{\mathrm{rel}}, g^{\mathrm{irr}} \equiv \{0\})$ obey the relation

$$\hat{S}_i(\boldsymbol{q}, \hat{g}_{\mathrm{E}}, \hat{g}_{\mathrm{h}}) = \lambda_i^{-1}\lambda_{\mathrm{h}}^2 S_i(\boldsymbol{q}, \lambda_{\mathrm{E}} g_{\mathrm{E}}, \lambda_{\mathrm{h}} g_{\mathrm{h}}) \tag{4.18}$$

with the arbitrary scale factors λ_i. The function $\hat{S}_i$ is related to a new RG generator $\hat{\mathscr{G}}$.

We consider the case of zero magnetic field, i.e. $g_{\mathrm{h}} = 0$, and write down the first three leading contributions of the spin–spin correlation

$$\langle S_{\boldsymbol{q}} S^{-\boldsymbol{q}} \rangle \approx q^{-2+\eta}(A + Bq^{-1/\nu}\tau + C_{\pm}(q^{-1/\nu}|\tau|)^{1-\alpha} + \ldots)V \tag{4.19}$$

with the critical exponents $\alpha = 2 - d/y_{\mathrm{E}}$, $\nu = 1/y_{\mathrm{E}}$ and the coefficients $A = s_{0\varnothing}(\boldsymbol{e})$, $B = s_{0\{\mathrm{E}\}}(\boldsymbol{e})$, $C_{\pm} = Cc'_{\pm}$, $C = s_{\mathrm{E}\varnothing}(\boldsymbol{e})$ and

$$c'_{\pm} = \frac{1}{V}\langle O_{\mathrm{E}}(g)\rangle|_{g^{\mathrm{irr}} \equiv \{0\}, g_{\mathrm{h}} = 0, g_{\mathrm{E}} = \pm 1} \tag{4.20}$$

(4.19) is in agreement with (1.1). Because of another normalization of $S_{\boldsymbol{q}}$ the factor V does not appear in the expression (1.1) of Brézin *et al* (1974a, b, 1977). The universality relation (4.18) entails that the identity

$$\hat{A}^2/\hat{B}\hat{C} = A^2/BC \tag{4.21}$$

holds with $\hat{A} = \lambda_{\mathrm{h}}^2 A$, $\hat{B} = \lambda_{\mathrm{h}}^2 \lambda_{\mathrm{E}} B$ and $\hat{C} = \lambda_{\mathrm{h}}^2 C/\lambda_{\mathrm{E}}$.

5. Spin–spin correlation for the model of a non-trivial fixed point

In order to determine the coefficients A, B, C and the critical exponents α, η, ν in (4.19) by means of Wegner's RG methods one has to choose a fixed point. In Stützer (1977) we calculated a non-trivial fixed point at the dimensionality $\alpha = 4 - \epsilon$ in order ϵ:

$$H^* = H^{*(\mathrm{a})} + \tfrac{16}{3}\pi^2 c^2 O_4^{(\mathrm{a})}\epsilon + \mathrm{O}(\epsilon^2) \tag{5.1}$$

where $H^{*(\mathrm{a})}$ is the trivial fixed point (Wegner 1976):

$$H^{*(\mathrm{a})} = u_0^{*(\mathrm{a})} V + \tfrac{1}{2} c \int_{\boldsymbol{q}} q^2 \sigma(q) S_{\boldsymbol{q}} S_{-\boldsymbol{q}} \tag{5.2}$$

and

$$O_4^{(\mathrm{a})} = \frac{1}{4!} \mathrm{e}^{-\mathscr{Y}} \int_{\boldsymbol{q}_1 \ldots \boldsymbol{q}_4} \prod_{i=1}^{4} (\phi(q_i) S_{\boldsymbol{q}_i}) \delta^d(\boldsymbol{q}_1 + \ldots + \boldsymbol{q}_4) \tag{5.3}$$

with

$$\mathscr{Y} \equiv \int_{\boldsymbol{q}} \frac{1}{2cq^2\sigma(q)} \frac{\delta^2}{\delta S_{\boldsymbol{q}} \delta S_{-\boldsymbol{q}}} \tag{5.4}$$

$$\phi(q) = (cq^2 w(q) + w^{-1}(q))^{-1} \tag{5.5}$$

$u_0^{*(\mathrm{a})}$ is an uninteresting constant. c is an arbitrary positive constant. The function

$$\sigma(q) = (cq^2 + w^{-2}(q))^{-1} \tag{5.6}$$

is regular at q and decays proportional to q^2 for $q \to \infty$. The evaluation of the fixed-point equation for (5.1) was carried out by a method given by Wegner (1976). The result of Shukla and Green (1974) differs from (5.1) only in an operator $\frac{1}{2} c \int_{\boldsymbol{q}} q^2 \sigma(q) S_{\boldsymbol{q}} S_{-\boldsymbol{q}} \epsilon$. We did not get this operator because we use a different constant c to that of Shukla and Green (1974). The fixed point (5.1) leads to systems with normal critical points.

Using the eigenoperators for the trivial fixed point of Wegner (1976) we calculated the operators O_{E} and $O_{\mathrm{h}}(\boldsymbol{q}, \tau)$:

$$O_{\mathrm{E}} = \tfrac{1}{2} e^{-\nu} \int_{\boldsymbol{q}} \phi^2(q) S_{\boldsymbol{q}} S_{-\boldsymbol{q}}$$

$$- \tfrac{8}{9}\pi^2 c^2 \, \mathrm{e}^{-\nu} \int_{\boldsymbol{q}_1 \ldots \boldsymbol{q}_4} \sigma(q_1) \prod_{i=1}^{4} (\phi(q_i) S_{\boldsymbol{q}_i}) \delta^d(\boldsymbol{q}_1 + \ldots \boldsymbol{q}_4)\epsilon + \mathrm{O}(\epsilon^2) \qquad (5.7)$$

$$O_{\mathrm{h}}(\boldsymbol{q}, \tau) = w(q)(S_{\boldsymbol{q}} - \delta H(\tau)/\delta S_{-\boldsymbol{q}} + \mathrm{O}(\tau^2) \qquad (5.8)$$

with $H(\tau) = H^* + \tau O_4$ (see (4.1) with (2.16)) and the scaling exponent $y_{\mathrm{E}} = 2 - \tfrac{1}{3}\epsilon$. Then we obtained the well known values for the critical exponents

$$\alpha = \tfrac{1}{6}\epsilon + \mathrm{O}(\epsilon^2); \qquad \nu = \tfrac{1}{2} + \tfrac{1}{12}\epsilon + \mathrm{O}(\varepsilon^3), \qquad (5.9)$$

used the value $\eta = \tfrac{1}{54}\epsilon^2 + \mathrm{O}(\epsilon^3)$ (Wilson and Kogut 1974) and obtained $y_{\mathrm{h}} = 3 - \tfrac{1}{2}\epsilon$.

In order to derive the spin–spin correlation according to Wegner's procedure we started with the Hamiltonian

$$H_l = H^* + \tau \, \mathrm{e}^{y_{\mathrm{E}} l} O_{\mathrm{E}} + \sum_{+,-} \kappa_{\mathrm{h}}^{\pm} \, \mathrm{e}^{y_{\mathrm{h}} l} O_{\mathrm{h}}(\pm q \, \mathrm{e}^l, \tau \, \mathrm{e}^{y_{\mathrm{E}} l})$$

$$+ \kappa_{\mathrm{h}}^{+} \kappa_{\mathrm{h}}^{-} (\psi_1(q, l) + \tau \psi_2(q, l)) + \mathrm{O}(\kappa_{\mathrm{h}}^{\pm 2}, \tau^2; \epsilon^2). \qquad (5.10)$$

In contrast to equation (3.5) we neglected already at this point terms of order τ^2 and all redundant and irrelevant contributions which give only corrections to the leading scaling behaviour. With the initial condition $\psi_1(q, 0) = \psi_2(q, 0) = 0$ (cf (3.2)) we solved the RG equation for H_l:

$$\psi_1(q, l) = -\mathrm{e}^{dl}(\mathrm{e}^{2l}\sigma(q \, \mathrm{e}^l) - \sigma(q))V + \tfrac{16}{3}\pi^2 c^2 \, \mathrm{e}^{y_{\mathrm{E}} l}(\mathrm{e}^{4l}\sigma^2(q \, \mathrm{e}^l) - \sigma(q))O_{\mathrm{E}}\epsilon \qquad (5.11)$$

$$\psi_2(q, l) = \mathrm{e}^{dl}(1 - \tfrac{1}{3} l \epsilon)(\mathrm{e}^{4l}\sigma^2(q \, \mathrm{e}^l) - \sigma^2(q))V$$

$$- \tfrac{32}{3}\pi^2 c^2 \, \mathrm{e}^{y_{\mathrm{E}} l}[\mathrm{e}^{2l}(\mathrm{e}^{4l}\sigma^2(q \, \mathrm{e}^l) - \sigma^2(q)) + (\mathrm{e}^{6l}\sigma^3(q \, \mathrm{e}^l) - \sigma^3(q))]O_{\mathrm{E}}\epsilon. \qquad (5.12)$$

In the limit $l \to \infty$ we have $(q \, \mathrm{e}^l)^{2n}\sigma^n(q \, \mathrm{e}^l) \to 1$ $(n = 1, 2, \ldots)$ and $\phi(q \, \mathrm{e}^l) \to 0$. As predicted the operators $O_j(\boldsymbol{q}_j \, \mathrm{e}^l, g \, \mathrm{e}^{yl})$ vanish in the limit, because their kernels contain the function $\phi(|\boldsymbol{q}_1 + \ldots + \boldsymbol{q}_{j-1} - \boldsymbol{q}_j \, \mathrm{e}^l|)$. The condition (3.6) is not true for the coefficients of the operators V and O_{E} in ψ_2. As for the term proportional to $l \, \mathrm{e}^{dl}$ the reason is that logarithmic contributions of $q \, \mathrm{e}^l$ appear in the derivation of ψ_2. In § 3, however, we did not discuss such logarithmic contributions. If we apply the RG transformation $\mathcal{R}_{-\tilde{l}}$ to $H_{\tilde{l}}$ with sufficiently large $\tilde{l}$, one can show that we get the Hamiltonian $\tilde{H}_0 = \mathcal{R}_{-\tilde{l}} H_{\tilde{l}}$ by replacing $\tilde{l}$ by zero. Then (3.9) gives the operator correlation function

$$\frac{1}{V}\langle O_{\mathrm{h}}(\boldsymbol{q}, \tau) O_{\mathrm{h}}(-\boldsymbol{q}, \tau)\rangle$$

$$= -\left(\sigma(q) - \frac{1}{cq^2}\right) + \left(\sigma^2(q) - \frac{1}{c^2 q^4}\right)\tau$$

$$+ \frac{16}{3}\pi^2 c^2 \left[\left(\sigma^2(q) - \frac{1}{c^2 q^4}\right)(1 - 2\tau) - 2\left(\sigma^3(q) - \frac{1}{c^3 q^6}\right)\tau\right] \frac{\langle O_{\mathrm{E}}\rangle}{V}\epsilon + \ldots. \qquad (5.13)$$

In order to be able to apply the relation (4.15) we determined the coefficients $b_{i\{E,E\}}$ in (2.18) for $i = 0$, E and obtained

$$O_E(\tau) = (1-2\tau)O_E - \frac{3}{16\pi^2 c^2 \epsilon}\tau V + O(\tau^2, \epsilon^2). \tag{5.14}$$

Equation (5.13) with (5.14) confirms the structure (3.10) of operator–operator correlation functions. The regular terms R_{ihh} contain the functions $\sigma^n(q)$. The singular contributions S_{ihh} are proportional to the powers q^{-2n} diverging in $q = 0$. From (5.9) we can see that the exponents -2 and -4 agree with the values of $-2+\eta$, $(-2+\eta)/\nu$ and $(-2+\eta)(1-\alpha)/\nu$ respectively in order ϵ^0. The regular functions $\sigma^n(q)$ are not universal, because they depend on the cut-off function θ. Furthermore they show no scaling behaviour.

From the equations (4.7), (4.17), (4.20), (5.13) with (5.14) it follows that the spin–spin correlation behaves like

$$\frac{1}{V}\langle S_q S_{-q}\rangle \approx \frac{1}{cq^2} - \frac{2}{c^2 q^4}\tau - \frac{16\pi^2}{3q^4}c_\pm\left(\frac{1}{q^4} - \frac{2}{cq^6}\tau\right)|\tau|^{1-\frac{1}{6}\epsilon} + \ldots \tag{5.15}$$

in the critical region, where $c_\pm = c'_\pm\epsilon$. $c'_\pm$ is a value of a functional integral (see (4.20)) which we did not calculate. One can estimate that $c'_\pm$ is of order ϵ^{-1}. Therefore, and since $b_{i\{E,E\}}$ is of order ϵ^{-1}, we obtain the correlation function (5.15) only in order ϵ^0.

The universality relation (4.18) requires that, in addition to the ratio in (4.21), the ratio AD/BC is independent of the scale factors λ_i, where $D = s_{E\{E\}}(e)$. Applying a transformation $C \to \hat{C}$ reflecting $\mathcal{G} \to \hat{\mathcal{G}}$ we get from (5.15) $\lambda_h = (C/\hat{C})^{1/2}$ and $\lambda_E = C/\hat{C}$ uniquely. The ratios mentioned before have the values

$$\frac{A^2}{BC} = \frac{3}{32\pi^2}, \qquad \frac{AD}{BC} = 1 \tag{5.16}$$

and are independent of the arbitrary constants C and $\hat{C}$ respectively. Thus (5.15) is universal.

We quote the result of Fisher and Aharony (1973) in order ϵ^0 and $\tau > 0$:

$$V^{-1}\langle S_q S_{-q}\rangle \approx c^F(q^{-2} - 2q^{-4}\tau + q^{-4}\tau^{1-\frac{1}{6}\epsilon} + \ldots) \tag{5.17}$$

with an arbitrary constant c^F. From this equation we find for the ratio in (4.21)

$$(A^F)^2/B^F C^F = -\tfrac{1}{2}c_+^F \tag{5.18}$$

with the constants A^F, B^F, C^F and c_+^F of Fisher. Because of universality the relation

$$c_+ = -3/16\pi^2 \tag{5.19}$$

must be true. With this value the spin–spin correlation (5.15) reads for $T > T_c$:

$$\frac{1}{V}\langle S_q S_{-q}\rangle \approx \frac{1}{cq^2} - \frac{2}{cq^4}\tau + \frac{1}{q^4}\tau^{1-\frac{1}{6}\epsilon} - \frac{2}{q^6}\tau^{2-\frac{1}{3}\epsilon} + \ldots. \tag{5.20}$$

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